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STATISTICAL PROPERTIES OF RADIATION FROM A GROUP OF ELECTRONS

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The statistical properties of a radiation spectrum from an electron bunch of arbitrary shape are considered. The kinetic equation for the probability density of the complex vector of total radiation field is derived and solved. The simple way to obtain the moments of the first two orders is presented. Average spectral intensity and its fluctuation are calculated without restrictions to the number of particles in the bunch.

1. Introduction

Nowadays there exists a great deal of interest in the generation of Terahertz radiation, which lies in the far-infrared region of spectrum [1, 2]. Currently the most powerful Terahertz source is the radiation of short electron bunches enhanced by coherence effects [3]. The theory of coherence enhancement of radiation was proposed in [4-7]. Using somewhat heuristic arguments it was shown, that the radiation spectrum contains a multi-particle coherence enhancement component, which is determined by the square of the Fourier transform of the longitudinal spatial distribution function of the electrons.

It was recognized in the early 50-s of the previous century that at high particle energies radiation becomes the dominant factor limiting the energy attainable with any magnetic accelerator. That is why the question on radiation produced by a group of electrons was raised by McMillan 60 years ago when the synchrotron concept was proposed [8, 9]. The first estimates of the radiation build-up due to coherence effects, when the radiation wavelength becomes less than the electron bunch length, showed the possible ways to eliminate it by a proper shielding [5, 6] . In [6] it was shown for the particular case of electrons symmetrically distributed around the mean position that the intensity enhancement is proportional to the Fourier component of the longitudinal particle density squared.

This initial stage of research resulted in the conclusion that the coherence effects do not contribute significantly to the energy losses in the accelerators being under development. Thus the whole problem was put on a shelf for a long period of time.

Since then, the main area of application of synchrotrons was shifted from high energy physics to the use as powerful sources of radiation. They are excellent light sources because of high brightness, high stability, clean environment, pulsed time structure. Incoherent Synchrotron radiation is now regarded as a standard light source.

The interest in the coherent enhancement of radiation in the far infrared was revived by the work of Michel [10] where very optimistic but incorrect qualitative theory was proposed. Anyway, this publication resulted in the extensive theoretical and experimental efforts having the goal to observe the coherence effects. Soon the initial conclusion was confirmed that the coherence effects are of negligible importance in the devices with long (1-100 cm) electron bunches [11, 12]. The negative experimental results motivated researchers to reject simple qualitative theory by Michel and to try to derive the result obtained in [6] more strictly. However, the old heuristic arguments were one more time applied in the proof of the above statement. It seemed so obscure for the authors [12] that they additionally verified their results with Monte-Carlo modeling to see if an average spectral intensity follows the predicted law.

The first successful observation of the coherence effect of Synchrotron Radiation was made in 1989 [13] when the bunch of 2 mm length was used. The intensity of coherent radiation was about 10^6 times stronger that could be expected for incoherent radiation at the wavelengths around 1 mm. The value of the enhancement factor was of the same order of magnitude as the number of electrons in the bunch. Quite soon the coherent Cherenkov and transition radiations were reported [14, 15] and the problem has gained much of practical importance.

The radiation produced by an electron bunch is stochastic in its very nature due to the shot noise which in turn is related to the discreteness of electron charge. First attempts to analyze the statistical properties of synchrotron radiation were done in [16-18]. The first and second – order correlations of complex Fourier harmonics of radiation field were found under the assumption that coherence effects can be neglected, and subsequently used to explain the experiments with a self-amplified spontaneous emission free electron laser at TESLA test facility at DESY [19].

Recent experimental advances in the generation of steady state, powerful coherent radiation have resulted in the growing area of application such as far infrared spectroscopy and terahertz imaging. All these applications require not only the knowledge of the radiation spectrum but the associated noise for the situation where coherence effects play significant role.

An accurate statistical description of the radiation spectrum requires the probability density distribution function to be found. If the bunch length is large than the wavelength of the emitted radiation, electrons radiate incoherently and the probability density distribution of spectral intensity is given by the well known Rayleigh or negative exponential distribution [20]. Probability distribution for the radiation spectrum of the bunch of arbitrary shape has not been analyzed till now. It is the aim of the present work to find the distribution function for this most general case. We restrict ourselves with the consideration of classical electromagnetic radiation, i.e. the noise associated with the quantum nature of light is neglected. In spite of that the theory we develop here is of common practical interest as it equally applies to the Cherenkov, transition and synchrotron radiation, as well as to the general problem of stochastic wave scattering [21].

In section 2 of the paper we formulate the general problem of shot noise in the electron bunch as related to the radiation spectrum. We show that this problem represents a particular case of the random walk in two dimensions. This means that well established methods, developed in the theory of Brownian motion [22], can be invoked for its solution. In section 3 we investigate the situation when the number of electrons in the bunch is not fixed but the probability of emission of an electron within a unit time interval is known. This case corresponds to the statistics of radiation in free electron lasers. The corresponding kinetic equation is derived and a simple procedure to obtain the moments of first two orders without its rigorous solution is presented. These two moments correspond to the average electric field phasor and the average spectral intensity. In section 4 the characteristic function for the probability density is found and the fourth–order moment, the fluctuation of intensity, is calculated. The simple analytic form for the distribution function of the complex vector of the total radiation field is presented in the limiting case of large number of particles, and some illustrative examples are provided. For the sake of brevity we analyze the case when electrons are distributed in one dimension along the line. The generalization of the results to multidimensional cases is obvious. In the Appendix similar results concerning the case of an electron bunch with a fixed number of particles are presented. This is the most typical situation for storage rings where bunch lifetime exceeds many hours.

2. The general problem of shot noise in the electron bunch

The general problem of radiation spectrum generated by a group of electrons can be described as follows. In the far zone the *k*-th electron produces a spectral component of electric field equal to

$$\vec{E}_k(\omega) = \vec{E}_0(\omega)\exp i\omega t_k \,, \qquad (1)$$

where t_k may be thought as the time moment when the *k*-th electron strikes a target in the case of transition radiation or as a temporal coordinate of the electron along the bunch in the case of synchrotron radiation. Note that the same problem arises in the radiation scattering when t_k is the temporal coordinate of a scattering atom. Thus the total electric field produced by the bunch of electrons can be defined as

$$\vec{E}(\omega) = \sum_k \vec{E}_k(\omega) = \vec{E}_0(\omega)\sum_k \exp i\omega t_k = \vec{E}_0(\omega)\big(X(\omega) + iY(\omega)\big), \qquad (2)$$

where the sum is taken over all the electrons in the bunch. Here we introduced an explicit definition for the real $X(\omega)$ and imaginary $Y(\omega)$ parts of the total complex sum. As usual, the power spectrum of radiation is given by the absolute value of the Pointing vector

$$J(\omega) = J_0(\omega)\left(X(\omega)^2 + Y(\omega)^2\right), \qquad (3)$$

where $J(\omega)$ is the spectral intensity of radiation and $J_0(\omega)$ is the spectrum radiated by a single particle . The spectrum emitted by the electron bunch is the product of the spectrum, radiated by a single particle, and a multi-particle factor. Now we require the probability $P(X,Y)dXdY$ that the components X,Y lie in the intervals $X, X+dX; Y, Y+dY$ provided the probability distribution for times t_k is known.

The way the problem is now defined makes it similar to a problem of the Brownian motion in two dimensions. Thus it can be solved by the general method of Markoff [22]. For the sake of clearness of the presentation we follow a slightly modified approach to the solution of such problem given by Rayleigh [23].

3. Fluctuating number of electrons in a bunch

Consider the bunch formed by the electrons emitted from a cathode so that the probability to emit an electron within a time interval $t, t+dt$ is $\alpha(t)dt$. The probability $\alpha(t)$ equals to the electric current of the bunch, divided by elementary electric charge. An average number of electrons, emitted during all the observation period equals to $N = \int \alpha(t)dt$. A schematic representation of the stochastic process of electron emission and the behavior of, say, $X(\omega)$ is shown in Fig.1.a)-c). In Fig.1a) we can see that the electrons are emitted with the probability per unit time $\alpha(t)$ having a bell – like shape. Dotted vertical lines show the time moments $t_1, t_2, t_3, ..., t_k$ when the emission acts take place. These moments were picked with a random number generator according to the shown probability distribution. Fig.1.b) shows the successive increments in X value and Fig.1c) demonstrates the resulting temporal behavior of X. The problem of practical importance is to find the limiting X, Y - point as time goes to infinity.

While looking at this picture one may ask a more general question. Namely, instead of searching for the probability of the final result we may look for the probability to observe the X, Y in the interval $X, X+dX; Y, Y+dY$ at an arbitrary time moment t.

There are two ways to reach the point X, Y at the moment $t+dt$. First of all, the system can already find itself in this point provided no electrons were emitted during the time interval $t, t+dt$. The second way requires a successive electron emitted during the time interval *dt*. In this case the only point from which the transition to X, Y may occur is the point with co-ordinates $X - cos(\omega t), Y - sin(\omega t)$.

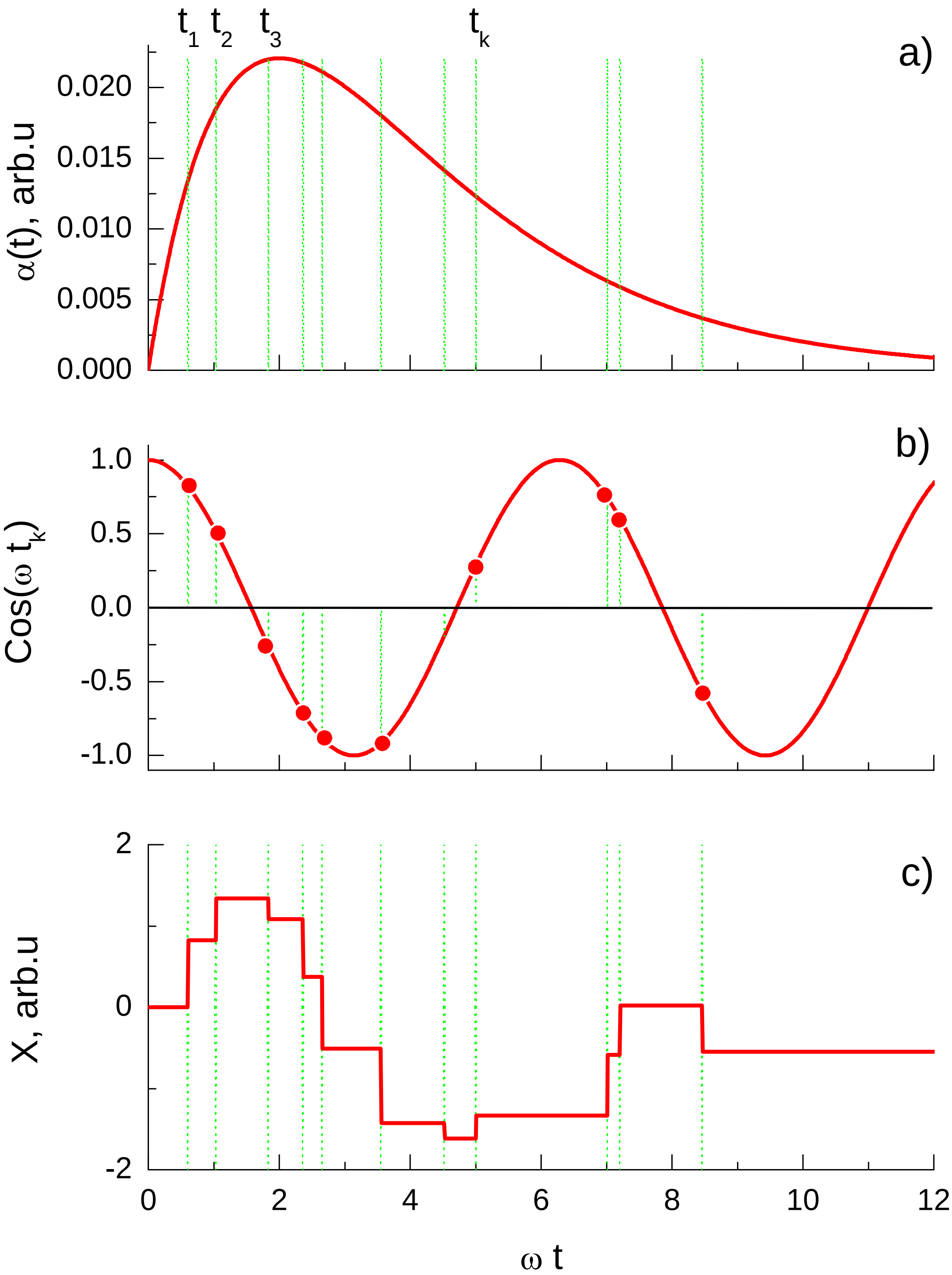

Fig.1. Schematic representation of a particular realization of the stochastic process

After these introductory remarks we can write the following kinetic equation for the probability distribution density

$$P(X,Y,\omega,t+dt) = P(X,Y,\omega,t)\big(1-\alpha(t)dt\big) + P\big(X-\cos(\omega t), Y-\sin(\omega t), \omega, t\big)\alpha(t)dt\,. \qquad (4)$$

Here $P(X,Y,\omega,t)$ is the probability density that determines the chances to find coordinates X,Y at the moment t. The first term in the right side of (4) corresponds to the first way to reach the point X,Y at the moment $t+dt$ without an electron emission and the second term is responsible for the transition when an electron is emitted during *d*t. In differential form the equation (4) reads as follows

$$\frac{\partial P(X,Y,\omega,t)}{\partial t} = -\alpha(t)P(X,Y,\omega,t) + \alpha(t)P\big(X-\cos(\omega t), Y-\sin(\omega t), \omega, t\big)\,. \qquad (5)$$

An initial condition for the probability density $P(X,Y,\omega,t)$ reflects the fact that there were no particles emitted in the infinitely long past, thus $X=0, Y=0$. It can be written as follows:

$$P(X,Y,\omega,-\infty) = \delta(X)\delta(Y)\,. \qquad (6)$$

Now we will consider the moments of the probability distribution, which can be obtained without a rigorous solution of (5).

A simple procedure to deduce the moments of the first two orders is described below. It is based on the general properties of the kinetic equation (5). The first and second order moments are usually defined as follows [20],

$$\langle X\rangle = \int XP(X,Y,\omega,t)dXdY;\ \langle Y\rangle = \int YP(X,Y,\omega,t)dXdY \qquad (7)$$

and

$$\left\langle X^2 \right\rangle = \int X^2 P(X,Y,\omega,t)dXdY; \ \left\langle Y^2 \right\rangle = \int Y^2 P(X,Y,\omega,t)dXdY . \tag{8}$$

Now in order to find the equation for the first-order moment, say $\langle X \rangle$, we multiply both sides of the equation (5) by X and then integrate it over the whole (*X*,*Y*) space. This results in the ordinary differential equations for the first-order moment

$$\frac{d\langle X \rangle}{dt} = \alpha(t)\cos(\omega t); \quad \frac{d\langle Y \rangle}{dt} = \alpha(t)\sin(\omega t) . \tag{9}$$

In deriving (9) we made an obvious transform of the co-ordinates X,Y to ξ,η and then applied the definition (7)

$$\int XP(X-\cos(\omega t), Y-\sin(\omega t),\omega,t)dXdY = \int(\xi+\cos(\omega t))P(\xi,\eta,\omega,t)d\xi d\eta = \langle X \rangle + \cos(\omega t) . \tag{10}$$

The equations for the second-order moments are as follows

$$\frac{d\left\langle X^2 \right\rangle}{dt} = \alpha(t)\cos^2(\omega t) + \frac{d\langle X \rangle^2}{dt} , \tag{11}$$

$$\frac{d\left\langle Y^2 \right\rangle}{dt} = \alpha(t)\sin^2(\omega t) + \frac{d\langle Y \rangle^2}{dt} . \tag{12}$$

Summing up equations (11) and (12) we obtain the multi-particle factor

$$\left\langle X^2 \right\rangle + \left\langle Y^2 \right\rangle = \int \alpha(t)dt + \langle X \rangle^2 + \langle Y \rangle^2 . \tag{13}$$

Following previous works [6, 12], the expression for the multi-particle factor can be re-written as follows

$$\langle X^2\rangle + \langle Y^2\rangle = N + \left|\int \alpha(t)\exp(i\omega t)dt\right|^2 . \tag{14}$$

The first term in the right side of (14) equal to the average number of electrons in the bunch N affects the incoherent input and the second term gives the enhancement of radiation due to coherence effect. A minor difference between the coherence term in (14) and the one derived previously [12] is that it is strictly proportional to N^2 but not to $N(N-1)$. This effect arises due to fluctuation of the number of electrons in a bunch itself, which was taken into account in our analysis. Previous theoretical treatment considered bunches with a fixed number of electrons only (see also Appendix).

Calculation of moments of higher orders with this method is too time-consuming. The most straight approach that can be applied to estimate their values is to use characteristic function method [20]. As the next step in our analysis we will find the characteristic function and estimate the intensity fluctuation

$$\langle \delta J^2(\omega)\rangle = \langle J^2(\omega)\rangle - \langle J(\omega)\rangle^2 . \tag{15}$$

It requires the fourth order moments to be calculated.

4. The probability distribution function, characteristic function

Now we solve (5) by taking Fourier transform of $P(X,Y,\omega,t)$ over spatial co-ordinates X,Y. The resulting characteristic function for $t\to\infty$ is given below

$$P(k,\lambda,\omega) = \frac{1}{(2\pi)^2}\exp-\left(\int_{-\infty}^{\infty} dt\alpha(t)\left(1-\exp\left(-ik\cos(\omega t)-i\lambda\sin(\omega t)\right)\right)\right). \tag{16}$$

The moments of any desired order can be found from (16) with the use of the well - known relation

$$\left\langle X^n Y^m \right\rangle = \frac{(2\pi)^2}{(-i)^{n+m}} \frac{\partial^{n+m}}{\partial^n k \partial^m \lambda} P(k,\lambda,\omega) \text{ with } k=0, \lambda=0 . \tag{17}$$

Substituting (16) into (17) and taking the derivatives we find the expression for the intensity fluctuation

$$\begin{aligned}
\frac{\left\langle \delta J^2(\omega)\right\rangle}{J_0^2(\omega)} = \Big[N + 4\left|\int dt\alpha(t)e^{i\omega t}\right|^2 + 2\left(\int dt\alpha(t)\cos^2\omega t\right)^2 + 2\left(\int dt\alpha(t)\sin^2\omega t\right)^2 \\
+4\int dt\alpha(t)\cos^2\omega t\left(\int dt\alpha(t)\cos\omega t\right)^2 + 4\int dt\alpha(t)\sin^2\omega t\left(\int dt\alpha(t)\sin\omega t\right)^2 + \\
4\left(\int dt\alpha(t)\sin\omega t\cos\omega t\right)^2 + 8\int dt\alpha(t)\cos\omega t\int dt\alpha(t)\sin\omega t\int dt\alpha(t)\sin\omega t\cos\omega t \Big]
\end{aligned} \tag{18}$$

There are two limiting cases of general interest. If the duration of the bunch τ is long compared to the period of oscillations $\omega\tau >> 1$, we deal with the completely incoherent case. Then equation (18) gives the following result

$$\frac{\left\langle \delta J^2(\omega)\right\rangle}{J_0^2(\omega)} = N^2 + N . \tag{19}$$

Thus, the deviation of intensity from the average value equals to this value when the source is incoherent, a result known since Rayleigh [23]. In the case of complete coherence the intensity fluctuation is as follows

$$\frac{\left\langle \delta J^2(\omega)\right\rangle}{J_0^2(\omega)} = 4N^3 + 6N^2 + N . \tag{20}$$

It is interesting to note, that contrary to the coherent emission from a bunch with a fixed number of particles (see Appendix), relation (20) predicts the noise in the coherent radiation due to the fluctuation of number of electrons in the bunch. Fig. 2, 3 illustrate the behavior of the average spectral intensity and its fluctuation, calculated for

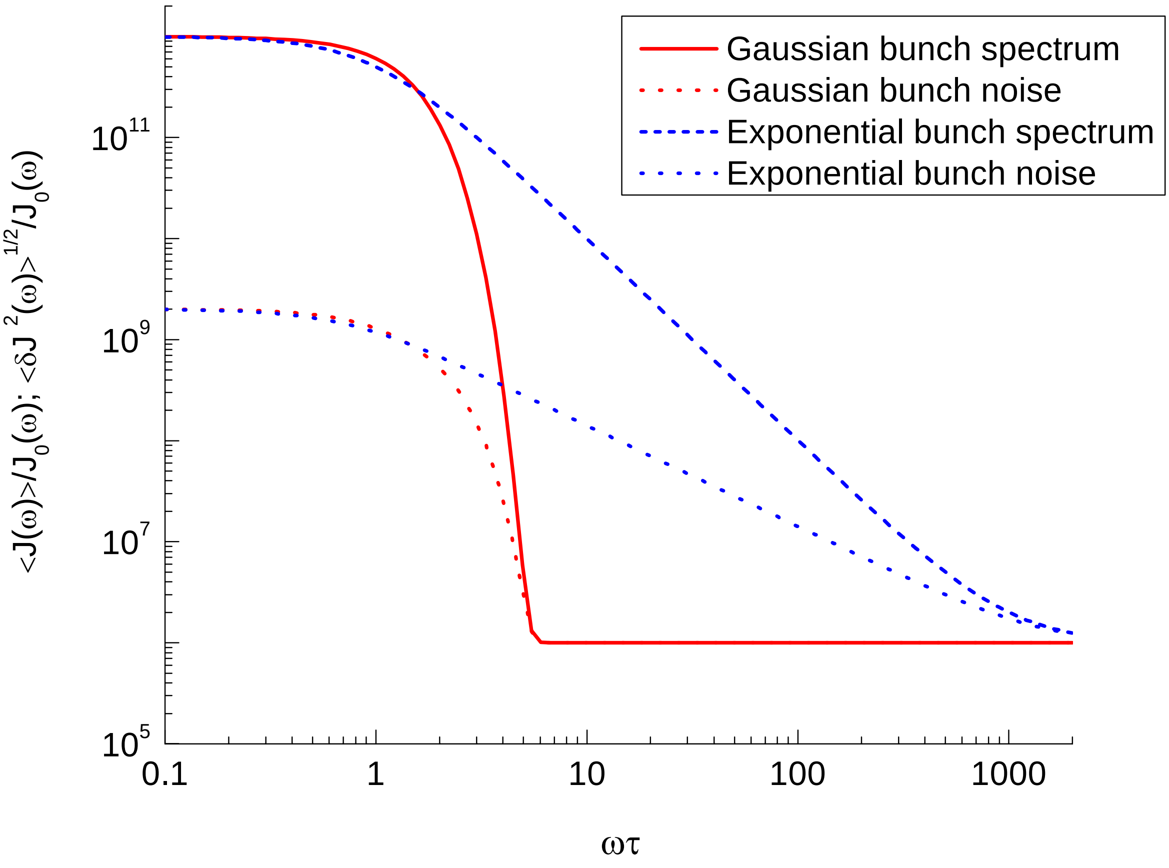

Fig.2. Average spectral intensity and its fluctuation
for bunches of different waveforms

two representative temporal bunch shapes – a Gaussian $\alpha(t)=\frac{N}{\sqrt{\pi}\tau}\exp-\frac{t^2}{\tau^2}$ and an exponential $\alpha(t)=\frac{N}{\tau}\exp-\frac{t}{\tau}$. Both bunches have equal number of particles $N=10^6$ but essentially different shape. We should note that the increase in steepness of the bunch shape appears naturally as a result of the self-impact on the bunch of its radiation forces [24]. Thus the exponential bunch with an idealized infinitely steep front may be used to estimate the upper limits of the radiation enhancement in the short wavelength spectral region. Fig.2. shows clearly that in both limiting cases of completely coherent and incoherent radiation the spectral intensity and its fluctuation do not depend on the

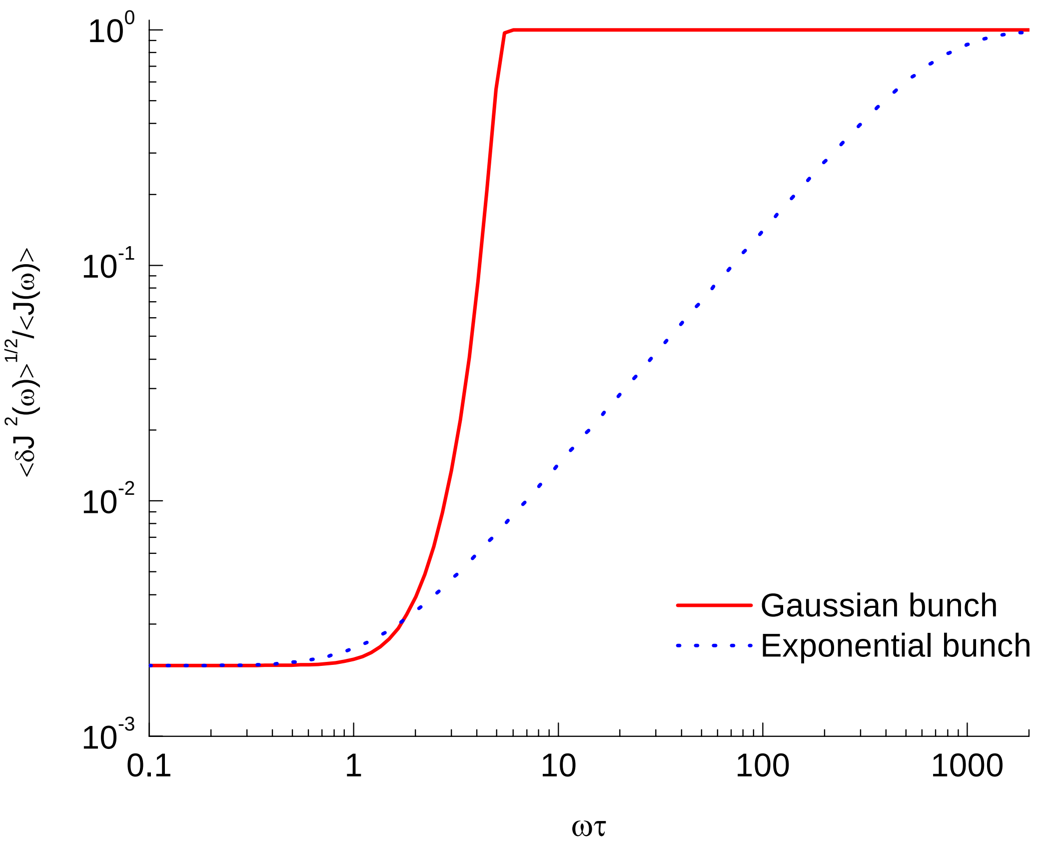

Fig.3. Noise to signal ratio for the bunches of different waveforms

bunch shape. At the same time the bunch shape plays a key role when the frequency range of radiation enhancement is of interest. The exponential bunch with its steep front has radiation enhancement which occupies the frequency range two orders of magnitude broader than the bunch with Gaussian shape. It leads to the possibility to produce powerful high frequency radiation with moderate total length bunches. The spectral behavior of noise to signal ratio for bunches of different shapes is shown in Fig.3. It is inversely proportional to the square root of the number of particles in the bunch in the low frequency coherent limit and goes to unity in the high frequency incoherent limit. It should be noted here that the noise to signal ratio depends not only on the number of particles but on the bunch shape as well. Thus the noise of the exponential bunch is larger than that from the Gaussian bunch provided we compare frequency regions where they emit equal intensities.

All the results presented above are valid for an arbitrary number of particles in the bunch. To get an analytic expression for the probability distribution function we have to consider the limiting case when the number of particles is large $N >> 1$. Fortunately, this limit represents the main practical interest. It is easily done by a standard method [22] with expansion of the characteristic function (16) into series. Then the result reads as follows

$$P(X,Y,\omega) = \frac{1}{2\pi\sqrt{4ab - c^2}} \exp - \left(\frac{b\left(X - \langle X \rangle\right)^2 + a\left(Y - \langle Y \rangle\right)^2 + c\left(X - \langle X \rangle\right)\left(Y - \langle Y \rangle\right)}{\left(4ab - c^2\right)} \right). \quad (21)$$

Here the coefficients are defined by the following relationships

$$\begin{aligned} &\langle X \rangle = \int_{-\infty}^{\infty} \alpha(t)\cos(\omega t)\,dt; \ \ \langle Y \rangle = \int_{-\infty}^{\infty} \alpha(t)\sin(\omega t)\,dt; \\ &a = \frac{1}{2}\int_{-\infty}^{\infty} \alpha(t)\cos^2(\omega t)\,dt;\ b = \frac{1}{2}\int_{-\infty}^{\infty} \alpha(t)\sin^2(\omega t)\,dt;\ c = \int_{-\infty}^{\infty} \alpha(t)\sin(\omega t)\cos(\omega t)\,dt. \end{aligned} \quad (22)$$

This probability density gives detailed information on the amplitude and the phase of the complex phasor of the radiation field. The distribution of the intensity is as follows

$$\begin{aligned} &P(I,\omega) = \frac{1}{4\pi\sqrt{4ab - c^2}} \int_0^{2\pi} \exp - F(\psi)\,d\psi \\ &F(\psi) = \frac{b\left(\sqrt{IG}\cos\psi - \langle X \rangle\right)^2 + a\left(\sqrt{IG}\sin\psi - \langle Y \rangle\right)^2 + c\left(\sqrt{IG}\cos\psi - \langle X \rangle\right)\left(\sqrt{IG}\sin\psi - \langle Y \rangle\right)}{\left(4ab - c^2\right)} \end{aligned} \quad (23)$$

Here we have introduced $I = J(\omega)/\langle J(\omega) \rangle$, the ratio of the intensity to the average intensity, and $G = \langle X^2 \rangle + \langle Y^2 \rangle$, given by (14).

The probability distribution (23) takes an especially simple form if the radiation is incoherent. Actually, in this case $\langle X \rangle = 0;\ \langle Y \rangle = 0;\ a = b = \frac{N}{4};\ c = 0;\ \ G = N$, then (23) gives Rayleigh distribution

$$P(I) = e^{-I}. \tag{24}$$

In the general case the integration in (23) still has to be done numerically.

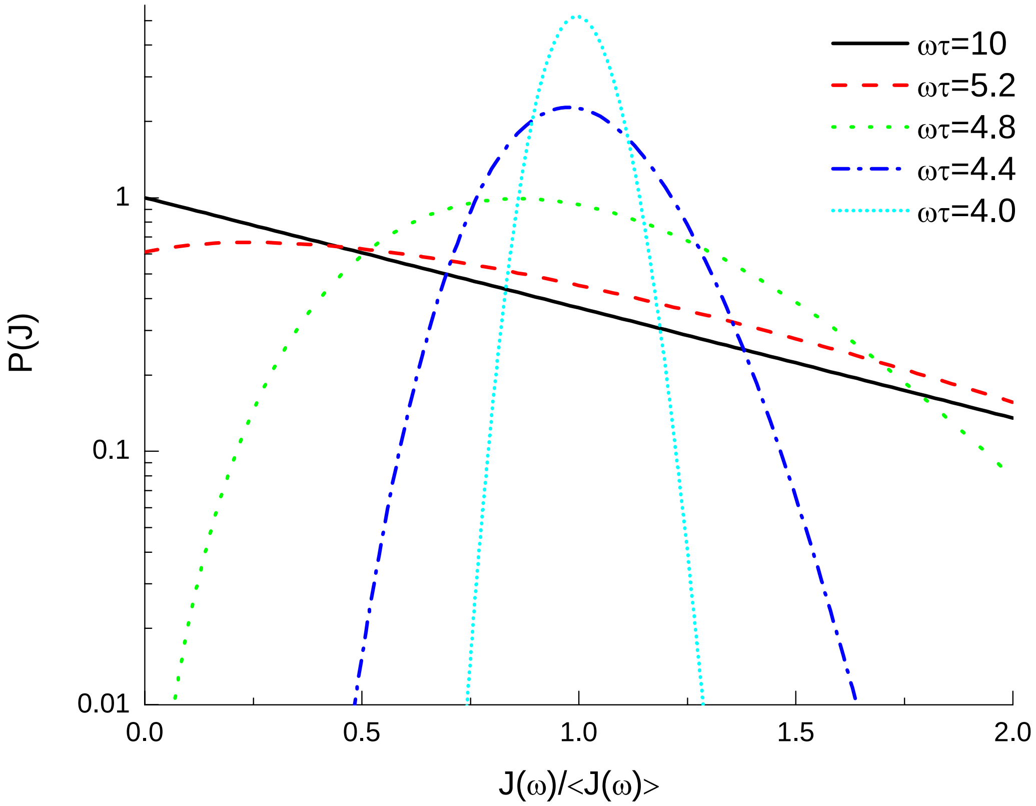

Fig.4. Intensity distribution function for Gaussian bunch waveform

Fig.4. gives an example of intensity distribution function, calculated for the Gaussian bunch shape in different spectral regions. In the range of high frequencies where radiation is incoherent it reproduces the Rayleigh distribution. As the coherence effects begin to prevail, the maximum of the distribution shifts towards the average intensity and the distribution width decreases. It is interesting to note that even in the region where the enhancement of radiation due to coherence is strongly pronounced the width of the distribution is still quite large. The lower the frequency the closer the intensity distribution converges to the Gaussian one.

Conclusion

The kinetic equation for the probability density of the complex vector of total radiation field emitted by an electron bunch of arbitrary shape has been formulated, and its general solution has been found. The results can be easily generalized to multi-dimension cases. We also described both the spectrum and noise properties of the radiation in the whole range of conditions, from coherent to incoherent case. As a result, the problem has reached the status of a mathematical theorem rather than a semi-empirical formula based on heuristic arguments. Our results show that the statistical behavior of the radiation may differ substantially from the results one could get using the existing approach. The results will be useful for the analysis of high-power sources of microwave, terahertz and X ray radiation based on pulsed electron beams.

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Appendix

For the sake of completeness here we present some results concerning the electron bunch with a fixed number of particles. Some of them are already obvious. For example, in the case of complete coherence there will be no noise while in the incoherent case the radiation of the bunch with a fixed number of particles should exhibit the same properties as the radiation from the fluctuating bunch.

Now the problem is formulated as follows. We look for the probability distribution function for the field phasor X,Y provided the bunch is composed of N particles and the probability to observe a particle within a time interval $t, t+dt$ equals to $f(t)dt$. The consideration similar to that presented in section 3 leads to the following equation which governs the probability distribution function

$$P_{N+1}(X,Y) = \int dt P_N\left(X - \cos(\omega t), Y - \sin(\omega t)\right) f(t) . \tag{25}$$

Here $P_N(X,Y)$ is the probability density to observe the point X,Y in the bunch containing N electrons. The characteristic function for the probability density has the following form

$$P_N(k,\lambda) = \frac{1}{(2\pi)^2}\left(\int dt f(t) \exp - i\left(k\cos(\omega t) + \lambda \sin(\omega t)\right)\right)^N . \tag{26}$$

The moments of the distribution are found by taking the derivatives of (26) according to (17). It gives the following results for the average intensity and its fluctuation

$$\left\langle X^2 \right\rangle + \left\langle Y^2 \right\rangle = N + N(N-1)\left|\int f(t)\exp(i\omega t)dt\right|^2 \tag{27}$$

and

$$\frac{\left\langle \delta J^2(\omega)\right\rangle}{J_0^2(\omega)} = 4N(N-1)(N-2)\left[\int dtf(t)\cos^2\omega t\left(\int dtf(t)\sin\omega t\right)^2\right.$$
$$\left.+\int dtf(t)\sin^2\omega t\left(\int dtf(t)\sin\omega t\right)^2\right] + 2N(N-1)\left[\left(\int dtf(t)\cos^2\omega t\right)^2\right. \tag{28}$$
$$\left.+\left(\int dtf(t)\sin^2\omega t\right)^2\right] + 4N(N-1)\left(\int dtf(t)\cos\omega t\sin\omega t\right)^2$$
$$+8N(N-1)(N-2)\int dtf(t)\sin\omega t\int dtf(t)\cos\omega t + \int dtf(t)\sin\omega t\cos\omega t$$

The essential feature of (28) is that in the incoherent limit it gives

$$\frac{\left\langle \delta J^2(\omega)\right\rangle}{J_0^2(\omega)} = N^2 - N\ . \qquad (29).$$

In the coherent limit (28) gives the obvious result with zero noise.

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